Karatzas and Shreve Section 1.1

Evan Donald

June 15, 2020

Problem 1.1.5 Assume for each t that $X_t(\omega) - Y_t(\omega) = 0$ for $\omega \in A_t$ such that $P(A_t^C) = 0$ (that is, Y is a modification of X). Let $B = \mathbb{Q} \cap [0, \infty)$. For $t_k \in B$ let $A_k = \{\omega | X_{t_k}(\omega) - Y_{t_k}(\omega) = 0\}$. Note that $\cup A_k^C$ is a countable union of null sets and so is also a null set. Now let $\omega \in (\cup A_k^C)^C = \cap A_k$. Then $X_{t_k}(\omega) - Y_{t_k}(\omega) = 0$ for every rational number t_k.

Fix $\omega \in \cap A_k$. Each $t \in [0, \infty)$ is the limit of a sequence of rational numbers $\{t_k\}_{k=1}^{\infty}$ in (t, ∞) and since $\omega \in \cap A_k$ we have $X_{t_k}(\omega) - Y_{t_k}(\omega) = 0$ for each t_k. By right continuity we see that it must be that:

$$X_t(\omega) - Y_t(\omega) = \lim_{k \to \infty} X_{t_k}(\omega) - Y_{t_k}(\omega) = 0$$

Then for this ω we see that $X_t(\omega) = Y_t(\omega$ for all t. Since $\omega \in \cap A_k$ was chosen arbitrarily, this is true for all such $\omega \in \cap A_k$. Recall $(\cap A_k)^C$ is a null set, so we have $X_t(\omega) = Y_t(\omega)$ for all t, except on a null set of Ω. In other words:

$$P(X_t = Y_t \ \forall \ 0 \leq t < \infty) = 1$$

Exercise 1.1.7 Enough to show continuity at the rational numbers in $(0, t_0)$. Continuity everywhere in $(0, t_0)$ will follow from right continuity and fact that left hand limits exist.

Let s be any real number in $(0, t_0)$. $X_t(\omega)$ is right continuous at s, it will be continuous if the left hand limit $\lim_{t \to s^+} X_t(\omega) = X_s(\omega)$. We can represent this in set notation using the following:

$$A = \bigcap_{j=1}^{\infty} \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} \{\omega : |X_{s-\frac{1}{n}}(\omega) - X_s(\omega)| \leq \frac{1}{j}\}$$

Explanation: Fix j. Then $\cup \cap \{\omega : |X_{s-\frac{1}{n}}(\omega) - X_s(\omega)| \leq \frac{1}{j}\}$ gives all ω such that $|X_{s-\frac{1}{n}}(\omega) - X_s(\omega)|$ is eventually less that $\frac{1}{j}$. Taking the intersection over all j we have all

ω such that $|X_{s-\frac{1}{n}}(\omega) - X_s(\omega)|$ is eventually less than all $\frac{1}{j}$. Meaning, the ω such that $\lim_{t\to s^+} X_t(\omega) = X_s(\omega)$. So A is the event that X_t is continuous at s.

Clearly, $X_s \in \mathcal{F}_s^X$. And for each n, $X_{s-\frac{1}{n}} \in \mathcal{F}_s^X$ since $\mathcal{F}_{s-\frac{1}{n}}^X \subset \mathcal{F}_s^X$. Thus $|X_{s-\frac{1}{n}} - X_s| \in \mathcal{F}_s^X$. Then $\{\omega : |X_{s-\frac{1}{n}}(\omega) - X_s(\omega)| \leq \frac{1}{j}\} \in \mathcal{F}_s^X$ for each j. Since σ-algebras are closed under countable unions and intersections we have that $A \in \mathcal{F}_s^X \subset \mathcal{F}_{t_0}^X$.

We have shown the result for general $s \in (0, t_0)$, so it holds for rational numbers in particular. The advantage of using rational numbers is they are countable. So, the event B that X_t is continuous at all rational numbers is a countable intersection of sets in $\mathcal{F}_{t_0}^X$. So $B \in \mathcal{F}_{t_0}^X$.

Exercise 1.1.10

Problem 1.1.16 Let $\tau : \Omega \to [0,\omega) \times \Omega$ be defined as $\tau(\omega) = (T(\omega), \omega)$. Then τ is measurable since T is measurable (each component of τ is measurable). Now let $A \in \mathcal{B}(\mathbb{R}^d)$. Note that $X(T) = X \circ \tau$. The composition of measurable functions is measurable, thus $X(T)$ is measurable.

Problem 1.1.17 Let $B = \{X_T \in A\}$. Note $B^C = \{X_T \in A^C\}$. Similarly if $B = \{X_T \in A\} \cup \{T = \infty\}$, then $B^C = \{X_T \in A^C\} \cap \{T < \infty\} = \{X_T \in A^C\}$ (I think).

Now let $B_n = \{X_T \in A_n\}$ and $C_n = \{X_T \in A_n\} \cup \{T = \infty\}$. Then the union of a countable number of such sets is

$$\{\bigcup\{X_T \in A_n\}\} \cup \{T = \infty\}$$
$$= \{X_T \in \cup A_n\} \cup \{T = \infty\}$$

So $\mathcal{F}_T$ is closed under complements and countable unions. Thus it is a σ-algebra. And clearly $\mathcal{F}_T$ is a subset of $\mathcal{F}$.