Karatzas and Shreve Section 1.2

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Problem 1.2.2 Let $\mathcal{C}$ be the collection of subsets A of Ω such that $1_A(\omega_1) = 1_A(\omega_2)$. Note that $\mathcal{C}$ is a σ-algebra (simple to show). Assume $X_s(\omega_1) = X_s(\omega_2)$ for some s. Then $1_A(\omega_1) = 1_A(\omega_2)$ for all $A \in \sigma(X_s)$ so $\sigma(X_s) \subset \mathcal{C}$. Since $X_t(\omega_1) = X_t(\omega_2)$ for all $t \in [0, T(\omega_1)]$ have $\sigma(X_{T(\omega_1)}) \subset \mathcal{C}$. So $1_A(\omega_1) = 1_A(\omega_2)$ for all A in $\sigma(X_{T(\omega_1)})$. Note that $B = \{\omega : T(\omega) = T(\omega_1)\}$ in $\mathcal{F}^X_{T(\omega)}$ and $1_B(\omega_1) = 1$. So $1_B(\omega_2) = 1$ which implies $\omega_2 \in B$. Thus $T(\omega_1) = T(\omega_2)$ as well.

Problem 1.2.6 We wish to show that $\{H_\Gamma < t\} \in \mathcal{F}_t$ for all t. For $t = 0$, $\{H_\Gamma < 0\} = \emptyset$ so is in $\mathcal{F}_0$. Let $t > 0$. Consider the set $\{H_\gamma < t\}$.

$$\{H_\Gamma < t\} = \{\omega \in \Omega : H_\Gamma(\omega) < t\}$$
$$= \{\omega \in \Omega : \inf(s \geq 0 : X_s(\omega) \in \Gamma) < t\}$$

Let $\omega \in \{H_\Gamma < t\}$. By the definition of infimum, there is some $s < t$ such that $X_s(\omega) \in \Gamma$. Since Γ is open there is some open ball B_ϵ such that $X_s(\omega) \in B_\epsilon \subset \Gamma$. Since $X_t(\omega)$ is right continuous there is some rational number r such that $s \leq r < t$ and $X_r(\omega) \in B_\epsilon \subset \Gamma$.

In fact, we can see that $\omega \in \{H_\Gamma < t\}$ if and only if there is some rational number $r < t$ such that $X_r(\omega) \in \Gamma$. (We have shown one direction, other direction is trivial). Thus:

$$\{H_\Gamma < t\} = \bigcup_{r \in \mathbb{Q} \cap [0,t)} \{X_r \in \Gamma\}$$

If $\{X_r \in \Gamma\} \in \mathcal{F}_t$ for each r, we are done (because $\mathcal{F}_t$ is closed under countable unions). Note X_t is adapted so $\{X_r \in \Gamma\} \in \mathcal{F}_r \subset \mathcal{F}_t$.

Problem 1.2.7 We wish to show that $\{H_\Gamma \leq t\} \in \mathcal{F}_t$ for all t. Begin with $t = 0$.

$$\{H_\Gamma \leq 0\} = \{\omega \in \Omega : X_0(\omega) \in \Gamma\}$$
$$= \{X_0 \in \Gamma\}$$

Since X is adapted, $\{X_0 \in \Gamma\} \in \mathcal{F}_0$. (Does not rely on fact that Γ is closed).

Now assume $t > 0$. Let us instead show that $\{H_\Gamma \leq t\}^C = \{H_\Gamma > t\} \in \mathcal{F}_t$. We argue that for closed set Γ that

$$\{H_\Gamma > t\} = \bigcap_{s \leq t} \{X_s \in \Gamma^C\}$$

If $\omega \in \{H_\Gamma > t\}$ then it is certainly true that $X_s(\omega) \in \Gamma^C$ for all $s \leq t$, so $\{H_\Gamma > t\} \subset \cap\{X_s \in \Gamma^C\}$

Now assume $\omega \in \cap\{X_s \in \Gamma^C\}$. It is clear that $H_\Gamma(\omega) \not< t$. But is it possible that $X_t(\omega) \in \Gamma^C$ and $t = H_\Gamma(\omega)$? No. If $t = H_\Gamma(\omega)$ there are two possibilities. Either $X_t(\omega) \in \Gamma$(but have already seen assumed that $X_t(\omega) \in \Gamma^C$ so contradiction). OR, there is a sequence $\{t_n\}$ such that $t \leq t_n$, $t_n \to t$ and $X_{t_n}(\omega) \in \Gamma$ for each t_n. Then by continuity of sample paths $X_{t_n}(\omega) \to X_t(\omega)$. Thus $X_t(\omega)$ is a limit point of Γ. Since Γ is closed, $X_t(\omega) \in \Gamma$. Thus, $X_t(\omega) \in \Gamma \cap \Gamma^C$, again a contradiction. Hence $\cap_{s \leq t}\{X_s \in \Gamma^C\} \subset \{H_\Gamma > t\}$.

Now we just need to show $\cap_{s \leq t}\{X_s \in \Gamma^C\}$ is in fact a countable union of measurable sets. This follows directly from continuity of sample paths.

$$\bigcap_{s \leq t} \{X_s \in \Gamma^C\} = \bigcap_{r \in \mathbb{Q} \cap [0,t]} \{X_r \in \Gamma^C\}$$

And $\{X_r \in \Gamma^C\} \in \mathcal{F}_r \subset \mathcal{F}_t$ for each r by the fact that X_t is adapted.

Problem 1.2.10

i). We wish to show that $\{S + T \leq t\} \in \mathcal{F}_t$. This is equivalent to showing that $\{S + T > t\} \in \mathcal{F}_t$. Note that :

$$\{S + T > t\} = \{S \geq t, T > 0\} \cup \{S > 0, T \geq t\} \cup \{S < t, T < t, S + T > t\}$$

The first two sets on the right hand side are in $\mathcal{F}_t$ by our preliminary assumptions. Consider the last set.

Assume that $S, T < t$ and let $\{q_n\}_{n=1}^{\infty}$ be the set of all rational numbers in $[0, t)$. Note that

$$\{T > q_n\} = \cup_{k=m}^{\infty}\{T \geq q_n + \frac{1}{k}\}$$

where m can be chosen to be sufficiently large so that $q_n + \frac{1}{m} < t$. Then $\{T > q_n\} \in \mathcal{F}_t$ for each q_n. and so $\{S \geq t - q_n, T > q_n\} \in \mathcal{F}_t$.

Finally note that

$$\{S, T < t, S + T > t\} = \cup_{n=1}^{\infty}\{S \geq t - q_n, T > q_n\} \in \mathcal{F}_t$$

ii).

$$\{S+T>t\} = \{T>t\} \cup \{T<t, S \geq t\} \cup \{S, T<t, S+T>t\}$$

Recall if T is a stopping time it is also an optional time, so we can use the same work in previous part to show last set on right hand side is in $\mathcal{F}_t$.

Problem 1.2.13. Assume $A \in \mathcal{F}_T$, i.e $A \cap \{T \leq t\} \in \mathcal{F}_t$ for all t. Note

$$A^C \cap \{T \leq t\} = \{T \leq t\} \setminus \{A \cap \{T \leq t\}\}$$

The right hand side is in $\mathcal{F}_t$ be definition of $\mathcal{F}_T$. So $A^C \in \mathcal{F}_T$. Now let $A_n \in \mathcal{F}_T$ for each n. Note that

$$\left(\bigcup A_n\right) \cap \{T \leq t\} = \bigcup (A_n \cap \{T \leq t\})$$

Since $\mathcal{F}_t$ is closed under countable unions, $\cup A_n \in \mathcal{F}_T$. So $\mathcal{F}_T$ is closed under complements and countable unions, so it is a σ-algebra.

Problem 1.2.14 Wish to show that $\{S \leq t\} \in \mathcal{F}_t$ for all t. One equivalent definition of $S \in \mathcal{F}_T$ is that $\{S \leq s\} \in \mathcal{F}_T$ for all $s \in \mathbb{R}$. Note $\{S \leq s\} \in \mathcal{F}_T$ by definition means $\{S \leq s\} \cap \{T \leq t\} \in \mathcal{F}_t$ for all t. So $\{S \leq s\} \cap \{T \leq t\} \in \mathcal{F}_t$ for all s, t. In particular, let $s = t$. Then for all t we have:

$$\{S \leq t\} \cap \{T \leq t\} \in \mathcal{F}_t$$

But since $S \geq T$, $\{S \leq t\} \cap \{T \leq t\} = \{S \leq t\}$. Thus, $\{S \leq t\} \in \mathcal{F}_t$ for all t.

Problem 1.2.17

i). Recall definition of restriction to $\{T \leq S\}$. For all $A \in \mathcal{F}$, we take $A \cap \{T \leq S\}$. Note that since $\{T \leq S\} \in \mathcal{F}_T$ we have $A \cap \{T \leq S\} \in \mathcal{F}_T$ for all $A \in \mathcal{F}_T$. Similarly, $A \cap \{T \leq S\} \in \mathcal{F}_S$. Thus $A \cap \{T \leq S\} \in \mathcal{F}_{S \wedge T}$.

Want to show that $E[Z|\mathcal{F}_T] = E[Z|\mathcal{F}_{T\wedge S}]$ so pick a set A in larger σ-algebra, $\mathcal{F}_T$. Then for such an A (and on $\{T \leq S\}$) we have:

$$\begin{aligned} E[1_A 1_{T \leq S} E[Z|\mathcal{F}_T]] &= E[1_{A \cap \{T \leq S\}} E[Z|\mathcal{F}_T]] \\ &= E[1_{A \cap \{T \leq S\}} Z] \\ &= E[1_{A \cap \{T \leq S\}} E[Z|\mathcal{F}_{T \wedge S}]] \\ & E[1_A 1_{T \leq S} E[Z|\mathcal{F}_{T \wedge S}]] \end{aligned}$$

ii). Similar to previous, we can show that $E[Z|\mathcal{F}_S] = E[Z|\mathcal{F}_{T\wedge S}]$ on $\{S < T\}$. Why ? In previous, didn't use specific form of inequality ($T \leq S$ versus $T < S$), just used fact that these sets are in $\mathcal{F}_{T\wedge S}$.

Now to show (ii), will show it on the two disjoint sets $\{T \leq S\}$ and $\{S < T\}$. By (i) we see that

$$1_{\{T\leq S\}}E[E[Z|\mathcal{F}_T]|\mathcal{F}_{T\wedge S}] = E[1_{\{T\leq S\}}E[Z|\mathcal{F}_T]|\mathcal{F}_{T\wedge S}] \qquad since \;\; \{T \leq S\} \in \mathcal{F}_{T\wedge S}$$

$$= E[1_{\{T\leq S\}}E[Z|\mathcal{F}_{T\wedge S}]|\mathcal{F}_{T\wedge S}] \qquad by\ (i)$$

$$= 1_{\{T\leq S\}}E[Z|\mathcal{F}_{T\wedge S}] \qquad since \;\; \{T \leq S\} \in \mathcal{F}_{T\wedge S}$$

And on $\{S > T\}$ have something similar, by justification above.

Problem 1.2.19

Y_t should be progressively measurable by simply invoking Tonelli/Fubini Theorem, the assumptions just guarantee that this is appropriate.

Fix $t \in [0, \infty)$. Since $f(s, x)$ is measurable on $[0, t] \times \mathbb{R}$ and X_s is progressively measurable, we see that $f(s, X_s)$ is measurable. And since f is bounded we have $f \in L^1([0, t] \times \Omega)$. By Fubini, we can hold one component constant and get a measurable function from the integral. That is, for each fixed element of Ω we have:

$$h(\omega) = \int_0^t f(s, X_s(\omega))ds$$

is a measurable function of t. Thus,

$$Y_t(\omega) = \int_0^t f(s, X_s(\omega))ds$$

is $\mathcal{B}([0, t]) \times \mathcal{F}_t$ measurable for each t and so is progressively measurable.

Now consider Y_T. Note

$$Y_T = \int_0^{T(\omega)} f(s, X_s(\omega))ds$$

The fact that this is $\mathcal{F}_T$ measurable follows from a previous result.